EXPLICIT BRACKET IN THE EXCEPTIONAL SIMPLE LIE SUPERALGEBRA $\mathfrak{cvect}(0|3)_*$

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ABSTRACT. This note is devoted to a more detailed description of one of the five simple exceptional Lie superalgebras of vector fields, $\mathfrak{cvect}(0|3)_*$, a subalgebra of $\mathfrak{vect}(4|3)$. We derive differential equations for its elements, and solve these equations. Hence we get an exact form for the elements of $\mathfrak{cvect}(0|3)_*$. Moreover we realize $\mathfrak{cvect}(0|3)_*$ by "glued" pairs of generating functions on a (3|3)-dimensional periplectic (odd symplectic) supermanifold and describe the bracket explicitly.

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Introduction

V. Kac [3] classified simple finite-dimensional Lie superalgebras over $\mathbb{C}$. Kac further conjectured [3] that passing to infinite-dimensional simple Lie superalgebras of vector fields with polynomial coefficients we only acquire the straightforward analogues of the four well-known Cartan series: $\mathfrak{vect}(n)$, $\mathfrak{svect}(n)$, $\mathfrak{h}(2n)$ and $\mathfrak{k}(2n+1)$ (of all, divergence-free, Hamiltonian and contact vector fields, respectively, realized on the space of dimension indicated).

It soon became clear [4], [1], [5], [6] that the actual list of simple vectoral Lie superalgebras is much larger. Several new series were found.

Next, exceptional vectoral algebras were discovered [8], [9]; for their detailed description see [10], [2]. All of them are obtained with the help of a Cartan prolongation or a generalized prolongation, cf. [8]. This description is, however, not always satisfactory; a more succinct presentation (similar to the one via generating functions for the elements of $\mathfrak{h}$ and $\mathfrak{k}$) and a more explicit formula for their brackets is desirable.

The purpose of this note is to give a more lucid description of one of these exceptions, $\mathfrak{cvect}(0|3)_*$. In particular we offer a multiplication table for $\mathfrak{cvect}(0|3)_*$ that is simpler than previous descriptions, by use of "glued" pairs of generating functions for the elements of $\mathfrak{cvect}(0|3)_*$.

This note can be seen as a supplement to [10]. To be self-contained and to fix notations we introduce some basic notions in section 0.

Throughout, the ground field is $\mathbb{C}$.

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§0. Background

0.1. We recall that a superspace V is a $\mathbb{Z}/2$-graded space; $V = V_{\bar 0} \oplus V_{\bar 1}$. The elements of $V_{\bar 0}$ are called *even*, those of $V_{\bar 1}$ *odd*. When considering an element $x \in V$, we will always assume that x is homogeneous, i.e. $x \in V_{\bar 0}$ or $x \in V_{\bar 1}$. We write $p(x) = \bar i$ if $x \in V_{\bar i}$. The superdimension of V is $(n|m)$, where $n = \dim(V_{\bar 0})$ and $m = \dim(V_{\bar 1})$.

For a superspace V, we denote by $\Pi(V)$ the same superspace with the shifted parity, i.e., $\Pi(V_{\bar i}) = V_{\bar i + \bar 1}$.

0.2. Let $x = (u_1, \dots, u_n, \xi_1, \dots, \xi_m)$, where $u_1, \dots, u_n$ are even indeterminates and $\xi_1, \dots, \xi_m$ odd indeterminates. In the associative algebra $\mathbb{C}[x]$ we have that $x \cdot y = (-1)^{p(x)p(y)} y \cdot x$ (by definition) and hence $\xi_i^2 = 0$ for all i. The derivations $\mathfrak{der}(\mathbb{C}[x])$ of $\mathbb{C}[x]$ form a Lie superalgebra; its elements are vector fields. These polynomial vector fields are denoted by $\mathfrak{vect}(n|m)$. Its elements are represented as

$$D = \sum_i f_i \frac{\partial}{\partial u_i} + \sum_j g_j \frac{\partial}{\partial \xi_j}$$

where $f_i \in \mathbb{C}[x]$ and $g_j \in \mathbb{C}[x]$ for all $i, j = 1..n$. We have $p(D) = p(f_i) = p(g_j) + \bar 1$ and the Lie product is given by the commutator

$$[D_1, D_2] = D_1 D_2 - (-1)^{p(D_1)p(D_2)} D_2 D_1.$$

On the vector fields we have a map, $\operatorname{div} : \mathfrak{vect}(n|m) \to \mathbb{C}[x]$, defined by

$$\operatorname{div} D = \operatorname{div}\Big(\sum_{i=1}^n f_i \frac{\partial}{\partial u_i} + \sum_{j=1}^n g \frac{\partial}{\partial \xi_j}\Big) = \sum_{i=1}^n \frac{\partial f_i}{\partial u_i} - (-1)^{p(D)} \sum_{j=1}^n \frac{\partial g_j}{\partial \xi_j}.$$

A vector field D that satisfies $\operatorname{div} D = 0$ is called *special*. The linear space of special vector fields in $\mathfrak{vect}(n|m)$ forms a Lie superalgebra, denoted by $\mathfrak{svect}(n|m)$.

0.3. Next we discuss the Lie superalgebra of Leitesian vector fields $\mathfrak{le}(n)$. It consists of the elements $D \in \mathfrak{vect}(n|n)$ that annihilate the 2-form $\omega = \sum_i du_i d\xi_i$. Hence $\mathfrak{le}(n)$ is an odd superanalogon of the Hamiltonian vector fields (in which case $\omega = \sum_i dp_i dq_i$). Similar to the Hamiltonian case, there is a map $\mathrm{Le} : \mathbb{C}[x] \to \mathfrak{le}(n)$, with $x = (u_1, \dots, u_n, \xi_1, \dots, \xi_n)$:

$$\mathrm{Le}_f = \sum_{i=1}^n \Big(\frac{\partial f}{\partial u_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial u_i}\Big)$$

Note that Le maps odd elements of $\mathbb{C}[x]$ to even elements of $\mathfrak{le}(n)$ and vice versa. Moreover $\mathrm{Ker}(\mathrm{Le}) = \mathbb{C}$. We turn $\mathbb{C}[x]$ (with shifted parity) into a Lie superalgebra with (Buttin) bracket $\{f, g\}$ defined by

$$\mathrm{Le}_{\{f,g\}} = [\mathrm{Le}_f, \mathrm{Le}_g]$$

A straightforward calculation shows that

$$\{f, g\} = \sum_{i=1}^n \Big(\frac{\partial f}{\partial u_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial u_i}\Big).$$

This way $\Pi\mathbb{C}[x]/\mathbb{C} \cdot 1$ is a Lie superalgebra isomorphic to $\mathfrak{le}(n)$. We call f the generating function of Le_f. *Here and throughout* $p(f)$ will denote the

parity in $\mathbb{C}[x]$, not in $\Pi\mathbb{C}[x]$. So $p(f)$ is the parity of the number of ξ in a term of f.

0.4. The algebra $\mathfrak{le}(n)$ contains certain important subalgebras. First of all there is $\mathfrak{sle}(n)$, the space of special Leitesian vector fields:

$$\mathfrak{sle}(n) = \mathfrak{le}(n) \cap \mathfrak{svect}(n|n).$$

We have seen that if $D \in \mathfrak{le}(n)$ then $D = \mathrm{Le}_f$ for some $f \in \mathbb{C}[x]$. Now $D \in \mathfrak{sle}(n)$ iff f is harmonic in the following sense

$$\Delta(f) := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial u_i \partial \xi_i} = 0$$

Usually we simply say $f \in \mathfrak{sle}(n)$, identifying f and Le_f. This Δ satisfies the condition $\Delta^2 = 0$ and hence $\Delta : \mathfrak{le}(n) \to \mathfrak{sle}(n)$. The image $\Delta(\mathfrak{le}(n)) =: \mathfrak{sle}^\circ(n)$ is an ideal of codimension 1 on $\mathfrak{sle}(n)$. This ideal, $\mathfrak{sle}^\circ(n)$, can also be defined by the exact sequence

$$0 \longrightarrow \mathfrak{sle}^\circ(n) \longrightarrow \mathfrak{sle}(n) \longrightarrow \mathbb{C} \cdot \mathrm{Le}_{\xi_1 \ldots \xi_n} \longrightarrow 0.$$

Note that if $\Phi = \sum u_i \xi_i$ and $f \in \mathfrak{sle}(n)$, then

$$\Delta(\Phi f) = (n + \deg_u f - \deg_\xi f) \cdot f$$

Let $\nu(f) = n + \deg_u f - \deg_\xi f$. Then $\nu(f) \neq 0$ iff $f \in \mathfrak{sle}^\circ(n)$. So on $\mathfrak{sle}^\circ(n)$ we can define the right inverse Δ^{-1} to Δ by the formula

$$\Delta^{-1} f = \frac{1}{\nu(f)} (\Phi f).$$

0.5. Cartan prolongs. We will repeatedly use Cartan prolongation. So let us recall the definition. Let $\mathfrak{g}$ be a Lie superalgebra and V a $\mathfrak{g}$-module. Set $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_0 = \mathfrak{g}$ and for $i > 0$ define the i-th *Cartan prolong* $\mathfrak{g}_i$ as the space of all $X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1})$ such that

$$X(w_0)(w_1, w_2, \dots, w_i) = (-1)^{p(w_0)p(w_1)} X(w_1)(w_0, w_2, \dots, w_i)$$

for all $w_0, \dots, w_i \in \mathfrak{g}_{-1}$.

The *Cartan prolong* (the result of Cartan's *prolongation*) of the pair $(V, \mathfrak{g})$ is $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \oplus_{i \geq -1} \mathfrak{g}_i$.

Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful. Then

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \subset \mathfrak{vect}(n|m) = \mathfrak{der}(\mathbb{C}[x]), \quad \text{where } n = \dim(V_{\bar{0}}) \text{ and } m = \dim(V_{\bar{1}})$$

and $x = (u_1, \dots, u_n, \xi_1, \dots, \xi_m)$. We have for $i \geq 1$

$$\mathfrak{g}_i = \{D \in \mathfrak{vect}(n|m) : \deg D = i, [D, X] \in \mathfrak{g}_{i-1} \text{ for any } X \in \mathfrak{g}_{-1}\}.$$

The Lie superalgebra structure on $\mathfrak{vect}(n|m)$ induces one on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$. This way the commutator of vector fields $[g, v]$, corresponds to the action $g \cdot v$, $g \in \mathfrak{g}$ and $v \in V$.

We give some examples of Cartan prolongations. Let $\mathfrak{g}_{-1} = V$ be an $(n|m)$-dimensional superspace and $\mathfrak{g}_0 = \mathfrak{gl}(n|m)$ the space of all endomorphisms of V. Then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(n|m)$. If one takes for $\mathfrak{g}_0$ only the supertraceless elements $\mathfrak{sl}(n|m)$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{svect}(n|m)$, the algebra of vector fields with divergence 0.

§1. The structure of $\mathfrak{vect}(0|3)_*$

1.1. In this note our primary interest is in a certain Cartan prolongation (denoted by $\mathfrak{vect}(0|3)_*$) and the extension $\mathfrak{cvect}(0|3)_*$ thereof. Here we will discuss $\mathfrak{vect}(0|3)_*$. Now $\mathfrak{vect}(0|3)_*$ is a short-hand notation for the Cartan prolongation with

$$V = \mathfrak{g}_{-1} = \Pi\Lambda(\eta_1, \eta_2, \eta_3)/\mathbb{C} \text{ and } \mathfrak{g}_0 = \mathfrak{der} V$$

So V is a superspace of dimension $(4|3)$, with

$$V_{\bar 0} = \langle \eta_1\eta_2\eta_3, \eta_1, \eta_2, \eta_3\rangle; \qquad V_{\bar 1} = \langle \eta_2\eta_3, \eta_3\eta_1, \eta_1\eta_2\rangle$$

and dim $\mathfrak{g}_0 = (12|12)$.

The elements of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ can be expressed as vector fields in $\mathfrak{vect}(4|3)$. Choosing

$$\eta_1\eta_2\eta_3 \simeq -\partial_y; \quad \eta_i \simeq -\partial_{u_i}; \quad \frac{\partial \eta_1\eta_2\eta_3}{\partial \eta_i} \simeq -\partial_{\xi_i}.$$

it is subject to straightforward verification that the elements of $\mathfrak{g}_0$, expressed as elements of $\mathfrak{vect}(4|3)$ are of the form:

$$\begin{array}{ll}
\partial_{\eta_1} \simeq -y\partial_{\xi_1} - \xi_2\partial_{u_3} + \xi_3\partial_{u_2} & -\eta_1\partial_{\eta_1} \simeq u_1\partial_{u_1} + \xi_2\partial_{\xi_2} + \xi_3\partial_{\xi_3} + y\partial_y \\
\partial_{\eta_2} \simeq -y\partial_{\xi_2} - \xi_3\partial_{u_1} + \xi_1\partial_{u_3} & -\eta_2\partial_{\eta_2} \simeq u_2\partial_{u_2} + \xi_1\partial_{\xi_1} + \xi_3\partial_{\xi_3} + y\partial_y \\
\partial_{\eta_3} \simeq -y\partial_{\xi_3} - \xi_1\partial_{u_2} + \xi_2\partial_{u_1} & -\eta_3\partial_{\eta_3} \simeq u_3\partial_{u_3} + \xi_1\partial_{\xi_1} + \xi_2\partial_{\xi_2} + y\partial_y
\end{array}$$

$$\begin{array}{lll}
\eta_1\partial_{\eta_2} \simeq -u_2\partial_{u_1} + \xi_1\partial_{\xi_2} & \eta_2\partial_{\eta_1} \simeq -u_1\partial_{u_2} + \xi_2\partial_{\xi_1} & \eta_1\eta_2\eta_3\partial_{\eta_1} \simeq -u_1\partial_y \\
\eta_2\partial_{\eta_3} \simeq -u_3\partial_{u_2} + \xi_2\partial_{\xi_3} & \eta_3\partial_{\eta_2} \simeq -u_2\partial_{u_3} + \xi_3\partial_{\xi_2} & \eta_1\eta_2\eta_3\partial_{\eta_2} \simeq -u_2\partial_y \\
\eta_3\partial_{\eta_1} \simeq -u_1\partial_{u_3} + \xi_3\partial_{\xi_1} & \eta_1\partial_{\eta_3} \simeq -u_3\partial_{u_1} + \xi_1\partial_{\xi_3} & \eta_1\eta_2\eta_3\partial_{\eta_3} \simeq -u_3\partial_y
\end{array}$$

$$\begin{array}{lll}
\eta_1\eta_2\partial_{\eta_3} \simeq -u_3\partial_{\xi_3} & \eta_1\eta_2\partial_{\eta_1} \simeq -u_1\partial_{\xi_3} - \xi_2\partial_y & \eta_1\eta_2\partial_{\eta_2} \simeq -u_2\partial_{\xi_3} + \xi_1\partial_y \\
\eta_2\eta_3\partial_{\eta_1} \simeq -u_1\partial_{\xi_1} & \eta_2\eta_3\partial_{\eta_2} \simeq -u_2\partial_{\xi_1} - \xi_3\partial_y & \eta_2\eta_3\partial_{\eta_3} \simeq -u_3\partial_{\xi_1} + \xi_2\partial_y \\
\eta_3\eta_1\partial_{\eta_2} \simeq -u_2\partial_{\xi_2} & \eta_3\eta_1\partial_{\eta_3} \simeq -u_3\partial_{\xi_2} - \xi_1\partial_y & \eta_3\eta_1\partial_{\eta_1} \simeq -u_1\partial_{\xi_2} + \xi_3\partial_y
\end{array}$$

1.2. Now we will give a more explicit description of $\mathfrak{vect}(0|3)_*$. It will turn out that $\mathfrak{vect}(0|3)_*$ is isomorphic to $\mathfrak{le}(3)$ as Lie superalgebra; however considered as $\mathbb{Z}$-graded algebras we have to define a different grading. The $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{le}(3;3)$ is $\mathfrak{le}(3)$ as Lie superalgebra with $\mathbb{Z}$-degree of D

$$D = \sum_i f_i \frac{\partial}{\partial u_i} + \sum_j g_j \frac{\partial}{\partial \xi_j}$$

the u-degree of f_i minus 1 (or the u-degree of g_j), i.e. deg $\xi_i = 0$.

Consider the map $i_1 : \mathfrak{le}(3;3) \to \mathfrak{vect}(4|3)$ given by

a.) If $f = f(u)$ then

$$i_1(\mathrm{Le}_f) = \mathrm{Le}_{\sum \frac{\partial f}{\partial u_i}\xi_j\xi_k - yf}$$

where y is treated as a parameter and $(i,j,k) \in A_3$ (even permutations of $\{1,2,3\}$).

b.) If $f = \sum f_i(u)\xi_i$ then

$$i_1(\mathrm{Le}_f) = \mathrm{Le}_f - \varphi(u)\sum \xi_i\partial_{\xi_i} + (-\varphi(u)y + \Delta(\varphi(u)\xi_1\xi_2\xi_3))\,\partial_y$$

where $\varphi(u) = \Delta(f)$ and Δ as given in section 0.4.

c.) If $f=\psi_1(u)\xi_2\xi_3+\psi_2(u)\xi_3\xi_1+\psi_3(u)\xi_1\xi_2$ then

$$i_1(\mathrm{Le}_f)=-\Delta(f)\partial_y-\sum_{i=1}^{3}\psi_i(u)\frac{\partial}{\partial\xi_i}.$$

d.) If $f=\psi(u)\xi_1\xi_2\xi_3$ then

$$i_1(\mathrm{Le}_f)=-\psi(u)\partial_y.$$

Note that i_1 preserves the $\mathbb{Z}$-degree. We have the following lemma.

1.3. Lemma. *The map i_1 is an isomorphism of $\mathbb{Z}$-graded Lie superalgebras between $\mathfrak{le}(3;3)$ and $\mathfrak{vect}(0|3)_* \subset \mathfrak{vect}(4|3)$.*

Proof. That i_1 is an embedding can be verified by direct computation. To prove that the image of i_1 is in $\mathfrak{vect}(0|3)_*$ it is enough to show that this is the case on the components $\mathfrak{le}(3;3)_{-1}\oplus\mathfrak{le}(3;3)_0$, i.e. on functions $f(u,\xi)$ of degree ≤ 1 with respect to u, as the Cartan prolongation is the biggest subalgebra $\mathfrak{g}$ of $\mathfrak{vect}(4|3)$, with given $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$. The proof that i_1 is surjective onto $\mathfrak{vect}(0|3)_*$ is given in corollary 4.6. □

A generalized version of Lemma 1.3 can be found in [10] and [7]. It states that $\mathfrak{le}(n;n)$ and $\mathfrak{vect}(0|n)_*$ are isomorphic for all $n\geq 1$.

§2. The construction of $\mathfrak{cvect}(0|3)_*$

2.1. Let us describe a general construction, which leads to several new simple Lie superalgebras. Let $\mathfrak{u}=\mathfrak{vect}(m|n)$, let $\mathfrak{g}=(\mathfrak{u}_{-1},\mathfrak{g}_0)_*$ be a simple Lie subsuperalgebra of $\mathfrak{u}$. Moreover suppose there exists an element $d\in\mathfrak{u}_0$ that determines an exterior derivation of $\mathfrak{g}$ and has no kernel on $\mathfrak{u}_+$. Let us study the prolong $\tilde{\mathfrak{g}}=(\mathfrak{g}_{-1},\mathfrak{g}_0\oplus\mathbb{C}d)_*$.

Lemma. *Either $\tilde{\mathfrak{g}}$ is simple or $\tilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathbb{C}d$.*

Proof. Let I be a nonzero graded ideal of $\tilde{\mathfrak{g}}$. The subsuperspace $(\mathrm{ad}\,\mathfrak{u}_{-1})^{k+1}a$ of $\mathfrak{u}_{-1}$ is nonzero for any nonzero homogeneous element $a\in\mathfrak{u}_k$ and $k\geq 0$. Since $\mathfrak{g}_{-1}=\mathfrak{u}_{-1}$, the ideal I contains nonzero elements from $\mathfrak{g}_{-1}$; by simplicity of $\mathfrak{g}$ the ideal I contains the whole $\mathfrak{g}$. If, moreover, $[\mathfrak{g}_{-1},\tilde{\mathfrak{g}}_1]=\mathfrak{g}_0$, then by definition of the Cartan prolongation $\tilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathbb{C}d$.

If, instead, $[\mathfrak{g}_{-1},\tilde{\mathfrak{g}}_1]=\mathfrak{g}_0\oplus\mathbb{C}d$, then $d\in I$ and since $[d,\mathfrak{u}_+]=\mathfrak{u}_+$, we derive that $I=\tilde{\mathfrak{g}}$. In other words, $\tilde{\mathfrak{g}}$ is simple. □

As an example, take $\mathfrak{g}=\mathfrak{svect}(m|n)$; $\mathfrak{g}_0=\mathfrak{sl}(m|n)$, $d=1_{m|n}$. Then $(\mathfrak{g}_{-1},\mathfrak{g}_0\oplus\mathbb{C}d)_*=\mathfrak{vect}(m|n)$.

2.2. Definition. *The Lie superalgebra $\mathfrak{cvect}(0|3)_*\subset\mathfrak{vect}(4|3)$ is the Cartan prolongation with $\mathfrak{cvect}(0|3)_{-1}=\mathfrak{vect}(0|3)_{-1}$ and $\mathfrak{cvect}(0|3)_0=\mathfrak{vect}(0|3)_0\oplus\mathbb{C}d$, with*

$$d=\sum u_i\partial_{u_i}+\sum\xi_i\partial_{\xi_i}+y\partial_y.$$

If now

$$f=\sum_{i=1}^{3}\xi_i\partial_{\xi_i}+2y\partial_y,$$

then it is clear that $f\in\mathfrak{vect}(0|3)\oplus\mathbb{C}d$, but $f\notin\mathfrak{vect}(0|3)$.

2.3. Theorem. *The Lie superalgebra* $\mathfrak{cvect}(0|3)_*$ *is simple.*

Proof. We know that $\mathfrak{vect}(0|3)_* \cong \mathfrak{le}(3;3)$ is simple. According to Lemma 2.1 it is sufficient to find an element $F \in \mathfrak{cvect}(0|3)_1$, which is not in $\mathfrak{vect}(0|3)_1$. For F one can take

$$F = y\xi_1\partial_{\xi_1} + y\xi_2\partial_{\xi_2} + y\xi_3\partial_{\xi_3} + y^2\partial_y - \xi_1\xi_2\partial_{u_3} - \xi_3\xi_1\partial_{u_2} - \xi_2\xi_3\partial_{u_1}$$

Indeed, one easily checks that $\partial_y F = f$, while

$$[\partial_{\xi_i}, F] = -\partial_{\eta_i} \qquad (i = 1, 2, 3),$$

and moreover $[\partial_{u_i}, F] = 0$. This proves the claim. □

Similar constructions are possible for general n. For $n = 2$ we obtain $\mathfrak{cvect}(0|2)_* \cong \mathfrak{vect}(2|1)$, while for $n > 3$ one can prove that $\mathfrak{cvect}(0|n)_*$ is not simple. For details, we refer to [10].

2.4. Lemma. *A vector field*

$$D = \sum_{i=1}^{3}(P_i\partial_{\xi_i} + Q_i\partial_{u_i}) + R\partial_y$$

in $\mathfrak{vect}(4|3)$ *belongs to* $\mathfrak{cvect}(0|3)_*$ *if and only if it satisfies the following system of equations:*

$$\frac{\partial Q_i}{\partial u_j} + (-1)^{p(D)}\frac{\partial P_j}{\partial \xi_i} = 0 \text{ for any } i \neq j; \tag{2.1}$$

$$\frac{\partial Q_i}{\partial u_i} + (-1)^{p(D)}\frac{\partial P_i}{\partial \xi_i} = \frac{1}{2}\left(\sum_{1\le j\le 3}\frac{\partial Q_j}{\partial u_j} + \frac{\partial R}{\partial y}\right) \text{ for } i = 1, 2, 3; \tag{2.2}$$

$$\frac{\partial Q_i}{\partial \xi_j} + \frac{\partial Q_j}{\partial \xi_i} = 0 \text{ for any } i, j; \text{ in particular } \frac{\partial Q_i}{\partial \xi_i} = 0; \tag{2.3}$$

$$\frac{\partial P_i}{\partial u_j} - \frac{\partial P_j}{\partial u_i} = -(-1)^{p(D)}\frac{\partial R}{\partial \xi_k} \tag{2.4}$$

for any k *and any even permutation* $\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}$.

$$\frac{\partial Q_i}{\partial y} = 0 \text{ for } i = 1, 2, 3; \tag{2.5}$$

$$\frac{\partial P_k}{\partial y} = (-1)^{p(D)}\frac{1}{2}\left(\frac{\partial Q_i}{\partial \xi_j} - \frac{\partial Q_j}{\partial \xi_i}\right) \tag{2.6}$$

for any k *and for any even permutation* $\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}$.

Proof. Denote by $\mathfrak{g} = \oplus_{i\ge -1}\mathfrak{g}_i$ the superspace of solutions of the system (2.1)–(2.6). Clearly, $\mathfrak{g}_{-1} \cong \mathfrak{vect}(4|3)_{-1}$. We directly verify that the images of the elements from $\mathfrak{vect}(0|3) \oplus \mathbb{C}d$ satisfy (2.1)–(2.6). Actually, we composed the system of equations (2.1)–(2.6) by looking at these images.

The isomorphism $\mathfrak{g}_0 = \mathfrak{vect}(0|3) \oplus \mathbb{C}d$ follows from dimension considerations.

Set

$$D_{u_j}(D)=\sum_{i\leq 3}(\frac{\partial P_i}{\partial u_j}\frac{\partial}{\partial \xi_i}+\frac{\partial Q_i}{\partial u_j}\frac{\partial}{\partial u_i})+\frac{\partial R}{\partial u_j}\frac{\partial}{\partial y};$$

$$D_{y}(D)=\sum_{i\leq 3}(\frac{\partial P_i}{\partial y}\frac{\partial}{\partial \xi_i}+\frac{\partial Q_i}{\partial y}\frac{\partial}{\partial u_i})+\frac{\partial R}{\partial y}\frac{\partial}{\partial y};$$

$$\tilde D_{\xi_j}(D)=(-1)^{p(D)}\sum_{i\leq 3}(\frac{\partial P_i}{\partial \xi_j}\frac{\partial}{\partial \xi_i}+\frac{\partial Q_i}{\partial \xi_j}\frac{\partial}{\partial u_i})+(-1)^{p(D)}\frac{\partial R}{\partial \xi_j}\frac{\partial}{\partial y}.$$

The operators D_{u_j}, D_y and $\tilde D_{\xi_j}$, clearly, *commute* with the $\mathfrak{g}_{-1}$-action. Observe: the operators *commute*, not *super*commute.

Since the operators in the equations (2.1)–(2.6) are linear combinations of only these operators D_{u_j}, D_y and $\tilde D_{\xi_j}$, the definition of Cartan prolongation itself ensures isomorphism of $\mathfrak{g}$ with $\mathfrak{cvect}(0|3)_*$. □

2.5. Remark. The left hand sides of eqs. (2.1)–(2.6) determine coefficients of the 2-form $L_D\omega$, where L_D is the Lie derivative and $\omega=\sum_{1\leq i\leq 3}du_id\xi_i$. It would be interesting to interpret the right-hand side of these equations in geometrical terms as well.

2.6. Remark. Lemma 2.4 illustrates how $\mathfrak{cvect}(0|3)_*$ can be characterized by a set of first order, constant coefficient, differential operators. This is a general fact of Cartan prolongations; one just replaces the linear constraints on $\mathfrak{g}_0$ by such operators. For example, for $\mathfrak{vect}(0|3)_*$ we have the equations (2.1)–(2.6) *and*

$$\frac{\partial R}{\partial y}-\sum_{i=1}^{3}\frac{\partial Q_i}{\partial u_i}=0 \tag{2.7}$$

Indeed, this equation is satisfied by all elements of $\mathfrak{vect}(0|3)_0$, see section 1.1, but not by d.

§3. Solution of differential equations (2.1) – (2.6)

Set $D^3_\xi=\frac{\partial^3}{\partial\xi_1\partial\xi_2\partial\xi_3}$.

3.1. Theorem. *Every solution of the system* (2.1) – (2.6) *is of the form:*

$$\begin{gathered}D=\mathrm{Le}_f+yA_f-(-1)^{p(f)}\left(y\Delta(f)+y^2D^3_\xi f\right)\partial_y+\\ A_g-(-1)^{p(g)}\left(\Delta(g)+2yD^3_\xi g\right)\partial_y,\end{gathered} \tag{3.1}$$

where $f,g\in\mathbb{C}[u,\xi]$ are arbitrary and the operator A_f is given by the formula:

$$A_f=\frac{\partial^2 f}{\partial\xi_2\partial\xi_3}\frac{\partial}{\partial\xi_1}+\frac{\partial^2 f}{\partial\xi_3\partial\xi_1}\frac{\partial}{\partial\xi_2}+\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\frac{\partial}{\partial\xi_3}. \tag{3.2}$$

Proof. First, let us find all solutions of system (2.1)–(2.6) for which $Q_1=Q_2=Q_3=0$. In this case the system takes the form

$$\frac{\partial P_j}{\partial\xi_i}=0 \text{ for } i\neq j \tag{2.1$'$}$$

$$(-1)^{p(D)}\frac{\partial P_i}{\partial\xi_i}=\frac{1}{2}\frac{\partial R}{\partial y} \text{ for } i=1,2,3 \tag{2.2$'$}$$

$$\frac{\partial P_i}{\partial u_j} - \frac{\partial P_j}{\partial u_i} = -(-1)^{p(D)} \frac{\partial R}{\partial \xi_k} \text{ for } (i,j,k) \in A_3 \tag{2.4'}$$

$$\frac{\partial P_k}{\partial y} = 0 \text{ for } k = 1,2,3 \tag{2.6'}$$

From (2.1′), (2.2′) and (2.6′) it follows that

$$P_i = \Psi_i(u_1,u_2,u_3) + \xi_i \varphi(u_1,u_2,u_3),$$

where $\varphi = \frac{1}{2}(-1)^{p(D)} \frac{\partial R}{\partial y}$. For brevity we will write $\Psi_i(u)$ and $\varphi(u)$. Then $R = (-1)^{p(D)} \cdot 2\varphi(u)y + R_0(u,\xi)$.

Let us expand the 3 equations of type (2.4′); their explicit form is:

$$\begin{aligned}
\frac{\partial R_0}{\partial \xi_1} &= -(-1)^{p(D)}(\frac{\partial \Psi_2}{\partial u_3} - \frac{\partial \Psi_3}{\partial u_2}) + (-1)^{p(D)}(\frac{\partial \varphi}{\partial u_2}\xi_3 - \frac{\partial \varphi}{\partial u_3}\xi_2),\\
\frac{\partial R_0}{\partial \xi_2} &= -(-1)^{p(D)}(\frac{\partial \Psi_3}{\partial u_1} - \frac{\partial \Psi_1}{\partial u_3}) + (-1)^{p(D)}(\frac{\partial \varphi}{\partial u_3}\xi_1 - \frac{\partial \varphi}{\partial u_1}\xi_3),\\
\frac{\partial R_0}{\partial \xi_3} &= -(-1)^{p(D)}(\frac{\partial \Psi_1}{\partial u_2} - \frac{\partial \Psi_2}{\partial u_1}) + (-1)^{p(D)}(\frac{\partial \varphi}{\partial u_1}\xi_2 - \frac{\partial \varphi}{\partial u_2}\xi_1).
\end{aligned}$$

The integration of these equations yields

$$\begin{aligned}
R_0 = (-1)^{p(D)}&(\Psi_0(u) - (\frac{\partial \Psi_2}{\partial u_3} - \frac{\partial \Psi_3}{\partial u_2})\xi_1 - \\
&(\frac{\partial \Psi_3}{\partial u_1} - \frac{\partial \Psi_1}{\partial u_3})\xi_2 - (\frac{\partial \Psi_1}{\partial u_2} - \frac{\partial \Psi_2}{\partial u_3})\xi_3 - (\frac{\partial \varphi}{\partial u_2}\xi_3\xi_1 + \frac{\partial \varphi}{\partial u_1}\xi_2\xi_3 + \frac{\partial \varphi}{\partial u_3}\xi_1\xi_2))\\
&= (-1)^{p(D)}(\Psi_0(u) + \Delta(-\Psi_1\xi_2\xi_3 - \Psi_2\xi_3\xi_1 - \Psi_3\xi_1\xi_2 - \varphi\xi_1\xi_2\xi_3)).
\end{aligned}$$

Therefore, any vector field D with $Q_1 = Q_2 = Q_3 = 0$ satisfying (2.1) – (2.6) is of the form

$$\begin{aligned}
D = \sum_{i=1}^{3} \Psi_i(u)\partial_{\xi_i} + \varphi(u)\sum_{i=1}^{3}\xi_i\partial_{\xi_i} + (-1)^{p(D)}&\\
\cdot (\Psi_0(u) + \Delta(-\Psi_1\xi_2\xi_3 - \Psi_2\xi_3\xi_1 - \Psi_3\xi_1\xi_2 - \varphi\xi_1\xi_2\xi_3) + 2\varphi(u)y)\partial_y.&
\end{aligned}$$

where, as before,

$$\Delta = \sum_{i=1}^{3} \frac{\partial}{\partial u_i}\frac{\partial}{\partial \xi_i}.$$

Set

$$g(u,\xi) = g_0(u,\xi) - \Psi_1\xi_2\xi_3 - \Psi_2\xi_3\xi_1 - \Psi_3\xi_1\xi_2 - \varphi\xi_1\xi_2\xi_3,$$

with $\Delta g_0 = \Psi_0$ and $\deg_\xi(g_0) \leq 1$. Then

$$A_g = \sum_{i=1}^{3}\Psi_i\partial_{\xi_i} + \varphi\sum_{i=1}^{3}\xi_i\partial_{\xi_i}; \quad D_\xi^3 g = \varphi \text{ and } (-1)^{p(D)} = (-1)^{p(g)+1}$$

for functions g homogeneous with respect to parity. In the end we get:

$$\begin{aligned}
D &= A_g + (-1)^{p(D)}(\Delta(g) + 2yD_\xi^3 g)\partial_y\\
&= A_g - (-1)^{p(g)}(\Delta(g) + 2yD_\xi^3 g)\partial_y.
\end{aligned} \tag{3.3}$$

Let us return now to the system (2.1) – (2.6). Equations (2.3), (2.5), (2.6) imply that there exists a function $f(u,\xi)$ (independent of y!) such that

$$Q_i = -(-1)^{p(D)}\frac{\partial f}{\partial \xi_i} \ \text{ for } \ i = 1,2,3.$$

Then (2.1) implies that

$$P_i = \frac{\partial f}{\partial u_i} + f_i(u,\xi_i,y).$$

From (2.6) it follows that

$$\frac{\partial f_i}{\partial y} = \partial_{\xi_j}\partial_{\xi_k} f \ \text{ for even permutations } \ (i,j,k)$$

or

$$f_i = y(\partial_{\xi_j}\partial_{\xi_k} f) + \tilde{P}_i(u,\xi_i).$$

Observe that $\tilde{P}_i$ satisfy (2.1′) and (2.6′); hence, in view of (2.2), $\frac{\partial \tilde{P}_i}{\partial \xi_i}$ does not depend on i. Therefore, we can choose $\tilde{R}$ so that $(\tilde{P}_i, \tilde{R})$ satisfy eqs. (2.1′), (2.2′), (2.4′), (2.6′). Thanks to the linearity of system (2.1) – (2.6) the vector field D is then of the form

$$D = D_f + \tilde{D}, \tag{3.4}$$

where D_f and $\tilde{D}$ are solutions of (2.1) – (2.6) such that $\tilde{D} = \sum \tilde{P}_i\partial_{\xi_i} + \tilde{R}\partial_y$ (i.e., $\tilde{D}$ is of the form (3.3)) and

$$\begin{aligned} D_f &= \sum(-(-1)^{p(D)}\frac{\partial f}{\partial \xi_i}\partial_{u_i} + \frac{\partial f}{\partial u_i}\partial_{\xi_i}) + \sum y(\partial_{\xi_j}\partial_{\xi_k} f)\partial_{\xi_i}) + R_f\cdot\partial_y \\ &= \mathrm{Le}_f + yA_f + R_f\partial_y. \end{aligned}$$

It remains to find R_f. Equation (2.2) takes the form

$$(-1)^{p(D)} y D_\xi^3 f = \frac{1}{2}(-(-1)^{p(D)}(\Delta f) + \frac{\partial R_f}{\partial y}).$$

Hence,

$$R_f = (-1)^{p(D)}(y^2 D_\xi^3 f + y\cdot(\Delta f) + R_0(u,\xi)).$$

Then, we can rewrite (2.4) as

$$-y\frac{\partial \Delta f}{\partial \xi_k} + \frac{\partial R_0}{\partial \xi_k} = y\partial_{u_j}\partial_{\xi_j}\partial_{\xi_k} f - y\partial_{u_i}\partial_{\xi_k}\partial_{\xi_i} f.$$

Observe that the right hand side of the last equation is equal to $-y\frac{\partial \Delta f}{\partial \xi_k}$. This means that $\frac{\partial R_0}{\partial \xi_k} = 0$ or $R_0 = R_0(u)$. Therefore, replacing $\tilde{R}$ with $\tilde{R} + R_0$ we may assume that $R_0 = 0$. Then

$$D_f = \mathrm{Le}_f + yA_f + (-1)^{p(D)}(y(\Delta f) + y^2 D_\xi^3 f)\partial_y. \tag{3.5}$$

By uniting (3.3) – (3.5) we get (3.1). □

§4 How to generate $\mathfrak{cvect}(0|3)_*$ by pairs of functions

We constructed $\mathfrak{cvect}(0|3)_*$ as an extension of $\mathfrak{vect}(0|3)_* \cong \mathfrak{le}(3;3)$, see lemma 1.3. Using the results of section 3, we obtain another embedding $i_2 : \mathfrak{le}(3) \to \mathfrak{vect}(0|3)_*$.

4.1. Lemma. *The map*

$$i_2 : \mathrm{Le}_f \to \mathrm{Le}_f + yA_f - (-1)^{p(f)} \left(y\Delta(f) + y^2 D^3_\xi f\right) \partial_y \tag{4.1}$$

determines an embedding of $\mathfrak{le}(3)$ *into* $\mathfrak{cvect}(0|3)_*$. *This embedding preserves the standard grading of* $\mathfrak{le}(3)$.

Proof. We have to verify the equality

$$i_2(\mathrm{Le}_{\{f,g\}}) = [i_2(\mathrm{Le}_f), i_2(\mathrm{Le}_g)].$$

Comparison of coefficients of different powers of y shows that the above equation is equivalent to the following system:

$$\mathrm{Le}_{\{f,g\}} = [\mathrm{Le}_f, \mathrm{Le}_g]. \tag{4.2}$$

$$A_{\{f,g\}} = [\mathrm{Le}_f, A_g] + [A_f, \mathrm{Le}_g] - (-1)^{p(f)}(\Delta(f)\cdot A_g + (-1)^{p(f)p(g)}\Delta(g)A_f). \tag{4.3}$$

$$[A_f, A_g] = (-1)^{p(f)} \left(D^3_\xi f \cdot A_g + (-1)^{p(f)p(g)} D^3_\xi g A_f\right). \tag{4.4}$$

$$\Delta(\{f,g\}) = \{\Delta f, g\} - (-1)^{p(f)}\{f, \Delta g\}. \tag{4.5}$$

$$\begin{aligned} D^3_\xi\{f,g\} = {} & \{D^3_\xi f, g\} - (-1)^{p(f)}\{f, D^3_\xi g\} - (-1)^{p(f)}(A_f(\Delta g) \\ & +(-1)^{p(f)p(g)} A_g(\Delta f)) + \Delta f D^3_\xi g - D^3_\xi f \Delta g. \end{aligned} \tag{4.6}$$

Equation (4.2) is known, see section 0.3. The equalities (4.3)–(4.6) are subject to direct verification. □

We found two embeddings $i_1 : \mathfrak{le}(3;3) \to \mathfrak{vect}(0|3)_*$ and $i_2 : \mathfrak{le}(3) \to \mathfrak{cvect}(0|3)_*$. Let us denote

$$\alpha_g = A_g - (-1)^{p(g)}(\Delta g + 2yD^3_\xi g)\partial_y.$$

We want to prove that the sum of the images of i_1 and i_2 cover the whole $\mathfrak{cvect}(0|3)_*$. According to Theorem 3.1, it is sufficient to represent α_g in the form $\alpha_g = i_1 g_1 + i_2 g_2$. For convenience we simply write f instead of Le_f.

4.2. Lemma. *For* α_g *we have:*

$$\alpha_g = \begin{cases} 0 & \textit{if } \deg_\xi g = 0 \\ i_1(-(\Delta g)\xi_1\xi_2\xi_3) & \textit{if } \deg_\xi g = 1 \\ i_1(g) & \textit{if } \deg_\xi g = 2 \\ i_1(-\Delta^{-1}(D^3_\xi g)) + i_2(\Delta^{-1}(D^3_\xi g)) & \textit{if } \deg_\xi g = 3. \end{cases}$$

The *right inverse* Δ^{-1} of Δ is given in section 0.4.
The proof of Lemma 4.2 is a direct calculation.

4.3. A wonderful property of $\mathfrak{sle}^\circ(3)$. In the standard grading of $\mathfrak{g} = \mathfrak{sle}^\circ(3)$ we have: $\dim \mathfrak{g}_{-1} = (3|3)$, $\mathfrak{g}_0 \cong \mathfrak{spe}(3)$. For the regraded superalgebra $R\mathfrak{g} = \mathfrak{sle}^\circ(3;3) \subset \mathfrak{le}(3;3)$ we have: $\dim R\mathfrak{g}_{-1} = (3|3)$, $R\mathfrak{g}_0 = \mathfrak{svect}(0|3) \cong \mathfrak{spe}(3)$. For the definition of $\mathfrak{spe}(3)$ we refer to [3] or [10]. Therefore, for $\mathfrak{sle}^\circ(3)$ and only for it among the $\mathfrak{sle}^\circ(n)$, the regrading R determines a nontrivial automorphism. In terms of generating functions the regrading is determined by the formulas:

1) $\deg_\xi(f) = 0$: $R(f) = \Delta(f\xi_1\xi_2\xi_3)$;
2) $\deg_\xi(f) = 1$: $R(f) = f$;
3) $\deg_\xi(f) = 2$: $R(f) = D_\xi^3(\Delta^{-1}f)$.

Note that $R^2(f) = (-1)^{p(f)+1}f$. Now we can formulate the following proposition.

4.4. Proposition. *The nondirect sum of the images of i_1 and i_2 covers the whole $\mathfrak{cvect}(0|3)_*$, i.e.,*

$$i_1(\mathfrak{le}(3;3)) + i_2(\mathfrak{le}(3)) = (\mathfrak{cvect}(0|3))_*.$$

We also have

$$i_1(\mathfrak{le}(3;3)) \cap i_2(\mathfrak{le}(3)) \cong \mathfrak{sle}^\circ(3;3) \cong \mathfrak{sle}^\circ(3).$$

Proof. The first part follows from Lemma 4.2. The second part follows by direct calculation from solving $i_2(\mathrm{Le}_f) = i_1(\mathrm{Le}_g)$. Note that $\mathrm{Le}_f \in \mathfrak{sle}^\circ(3)$ iff $\Delta(f) = 0$ and $D_\xi^3 f = 0$, and similar for $\mathrm{Le}_g \in \mathfrak{sle}^\circ(3;3)$. The equation $i_2(\mathrm{Le}_f) = i_1(\mathrm{Le}_g)$ is only solvable if $f \in \mathfrak{sle}^\circ(3)$ and $g \in \mathfrak{sle}^\circ(3;3)$, and in this case we obtain $g = (-1)^{p(f)+1}Rf$. $\square$

Therefore, we can identify the space of the Lie superalgebra $\mathfrak{cvect}(0|3)_*$ with the quotient space of $\mathfrak{le}(3;3) \oplus \mathfrak{le}(3)$ modulo

$$\{(-1)^{p(g)+1}Rg \oplus (-g), g \in \mathfrak{sle}^\circ(3)\}.$$

In other words, we can represent the elements of $\mathfrak{cvect}(0|3)_*$ in the form of the pairs of functions

$$(f,g), \quad \text{where} \quad f,g \in \Pi\mathbb{C}[u,\xi]/\mathbb{C}\cdot 1 \tag{4.7}$$

subject to identifications

$$(-1)^{p(g)+1}(Rg, 0) \sim (0, g) \quad \text{for any} \quad g \in \mathfrak{sle}^\circ(3).$$

4.5. Corollary. *The map φ defined by the formula*

$$\varphi|_{i_1(\mathfrak{le}(3;3))} = \text{sign } i_2 i_1^{-1}; \qquad \varphi|_{i_2(\mathfrak{le}(3))} = i_1 i_2^{-1}$$

is an automorphism of $\mathfrak{cvect}(0|3)_*$. Here $\text{sign}(D) = (-1)^{p(D)}D$.

The map φ may be represented in inner coordinates of $\mathfrak{vect}(4|3)$ as a regrading by setting $\deg y = -1; \deg u_i = 1; \deg \xi_i = 0$.

In the representation (4.7) we have

$$\varphi(f,g) = (g, (-1)^{p(f)+1}f).$$

Now we can complete the proof of Lemma 1.3.

4.6. Corollary. *The embedding $i_1 : \mathfrak{le}(3) \to \mathfrak{cvect}(0|3)_*$ is a surjection onto $\mathfrak{vect}(0|3)_*$.*

Proof. By Proposition 4.4 we merely have to prove that $i_2(\mathrm{Le}_f) \in \mathfrak{vect}(0|3)_*$ iff $\Delta f = 0$ and $D^3_\xi f = 0$. Applying equation (2.7) to $i_2(\mathrm{Le}_f)$, this follows immediately. $\square$

§5 The bracket in $\mathfrak{cvect}(0|3)_*$

Now we can determine the bracket in $\mathfrak{cvect}(0|3)_*$ in terms of representation (f, g) as stated in formula (4.7).

We do this via α_g. By Theorem 3.1 any $D \in \mathfrak{cvect}(0|3)_*$ is of the form $D = i_2(f) + \alpha_g$ for some generating functions f and g. To determine the bracket $[i_2(f), i_1(h)]$, we

1. Compute the brackets $[i_2 f, \alpha_g]$ for any $f, g \in \mathbb{C}[u, \xi]/\mathbb{C} \cdot 1$;
2. Represent $i_1(h)$ in the form

$$i_1(h) = i_2 a(h) + \alpha_{b(h)} \text{ for any } h \in \mathbb{C}[u, \xi]/\mathbb{C} \cdot 1; \tag{5.1}$$

In Lemma 4.2 we expressed α_g in i_1 and i_2.

Remark. The functions $a(h)$ and $b(h)$ above are not uniquely defined. Any representation will do.

5.1. Lemma. *For any functions $f, g \in \mathbb{C}[u, \xi]/\mathbb{C} \cdot 1$ the bracket $[i_2 f, \alpha_g]$ is of the form*

$$[i_2 f, \alpha_g] = i_2 F + \alpha_G, \tag{5.2}$$

where

$$F = f \cdot D^3_\xi g - (-1)^{(p(f)+1)(p(g)+1)} A_g f \qquad \textit{and} \qquad G = -f\Delta g$$

Proof. Direct calculation gives that

$$\begin{aligned}
[i_2 f, \alpha_g] = {} & [\mathrm{Le}_f, A_g] + (-1)^{p(f)p(g)+p(f)+1} \Delta g \cdot A_f \\
& + y\Big([A_f, A_g] + (-1)^{p(f)p(g)+p(f)+1} \cdot 2 \cdot D^3_\xi g \cdot A_f\Big) \\
& + (-1)^{p(g)+1}\Big(\{f, \Delta g\} + (-1)^{p(f)} \Delta f \cdot \Delta g\Big)\partial_y \\
& + \Big((-1)^{p(g)+1} A_f(\Delta g) + (-1)^{p(f)p(g)+p(g)+1} A_g(\Delta f) \\
& \quad + 2 \cdot (-1)^{p(g)+1}\{f, D^3_\xi g\} + 2 \cdot (-1)^{p(f)+p(g)+1} D^3_\xi f \cdot \Delta g\Big) y\partial_y \\
& + (-1)^{p(f)+p(g)+1} 2 \cdot D^3_\xi f \cdot D^3_\xi g \cdot y^2 \partial_y.
\end{aligned}$$

In order to find the functions F and G, it suffices to observe that the coefficient of ∂_y, non-divisible by y, should be equal to $(-1)^{p(G)+1}\Delta G$. This implies the equations:

$$(-1)^{p(G)+1}\Delta G = (-1)^{p(g)+1}\Big(\{f, \Delta g\} + (-1)^{p(f)} \Delta f \cdot \Delta g\Big)$$

or

$$(-1)^{p(G)+1}\Delta G = (-1)^{p(f)+p(g)+1}\Delta(f \cdot \Delta g).$$

Here $p(G) = p(f \cdot \Delta g) = p(f) + p(g) + 1$. Hence, $\Delta G = \Delta(-f\Delta g)$. Since G is defined up to elements from $\mathfrak{sle}^\circ(3)$, we can take $G = -f\Delta g$.

The function F to be found is determined from the equation

$$i_2 F = [i_2 f, \alpha_g] - \alpha_G. \tag{5.3}$$

By comparing the coefficients of $y\partial_y$ in the left and right hand sides of (5.3) we get

$$\begin{aligned}(-1)^{p(F)+1}\Delta F = &(-1)^{p(g)+1}A_f(\Delta g) + (-1)^{p(f)p(g)+p(g)+1}A_g(\Delta f)\\ &+ 2(-1)^{p(g)+1}\{f, D^3_\xi g\} + (-1)^{p(f)+p(g)+1}2\cdot D^3_\xi f\cdot\Delta g\\ &- 2\cdot(-1)^{p(f)+p(g)}D^3_\xi(-f\Delta g).\end{aligned}$$

Observe that

$$\begin{aligned}D^3_\xi(f\Delta g) &= (D^3_\xi f)\Delta g + (-1)^{p(f)}A_f(\Delta g) + \sum_{i=1}^3 \frac{\partial f}{\partial\xi_i}\frac{\partial}{\partial u_i}(D^3_\xi g)\\ &= (D^3_\xi f)\cdot\Delta g + (-1)^{p(f)}A_f(\Delta g) + (-1)^{p(f)}\{f, D^3_\xi g\}.\end{aligned}$$

Then

$$(-1)^{p(F)+1}(\Delta F) = (-1)^{p(g)}A_f(\Delta g) + (-1)^{p(f)p(g)+p(g)+1}A_g(\Delta f).$$

By comparing parities we derive that

$$p(F)+1 = p(A_f(\Delta g)) = p(f)+1+p(g)+1 = p(f)+p(g).$$

It follows that

$$\Delta F = (-1)^{p(f)}A_f(\Delta g) + (-1)^{p(f)p(g)+p(f)+1}A_g(\Delta f).$$

Let us transform the right hand side of the equality obtained. The sums over i, j, k are over $(i, j, k)\in A_3$:

$$\begin{aligned}&(-1)^{p(f)}A_f(\Delta g) + (-1)^{p(f)p(g)+p(f)+1}A_g(\Delta f)\\ =&\sum(-1)^{p(f)}\partial_{\xi_j}\partial_{\xi_k}f\cdot\partial_{\xi_i}(\sum_{s=1}^3\partial_{u_s}\partial_{\xi_s}g)\\ &+ (-1)^{p(f)p(g)+p(f)+1}\sum\partial_{\xi_j}\partial_{\xi_k}g\partial_{\xi_i}(\sum_{s=1}^3\partial_{u_s}\partial_{\xi_s}f)\\ =&(-1)^{p(f)p(g)+p(f)}\cdot\sum((\partial_{u_j}\partial_{\xi_i}\partial_{\xi_j}g + \partial_{u_k}\partial_{\xi_i}\partial_{\xi_k}g)\cdot\partial_{\xi_j}\partial_{\xi_k}f)\\ &- (-1)^{p(f)p(g)+p(f)}\cdot\sum(\partial_{\xi_j}\partial_{\xi_k}g(\partial_{u_j}\partial_{\xi_i}\partial_{\xi_j}f + \partial_{u_k}\partial_{\xi_i}\partial_{\xi_k}f))\\ =&(-1)^{p(f)p(g)+p(f)}\sum\partial_{u_k}(\partial_{\xi_i}\partial_{\xi_k}g\cdot\partial_{\xi_j}\partial_{\xi_k}f + \partial_{\xi_j}\partial_{\xi_k}g\cdot\partial_{\xi_k}\partial_{\xi_i}f)\\ =&(-1)^{p(f)p(g)+p(f)+p(g)}\sum_{k=1}^3(\partial_{u_k}\partial_{\xi_k}(A_g f) - \partial_{u_k}D^3_\xi g\cdot\partial_{\xi_k}f))\\ =&-(-1)^{(p(f)+1)(p(g)+1)}\Delta(A_g f) + (-1)^{p(f)p(g)+p(f)}\Delta(D^3_\xi g\cdot f)\\ =&\Delta(f\cdot D^3_\xi g) - (-1)^{(p(f)+1)(p(g)+1)}\Delta(A_g f).\end{aligned}$$

Then

$$F = f\cdot D^3_\xi g - (-1)^{(p(f)+1)(p(g)+1)}A_g f + F_0,\quad\text{where}\quad\Delta F_0 = 0.$$

We have shown how to *find* functions F and G. To prove Lemma 5.1 it only remains to compare the elements of the same degree in y in the right-hand and the left-hand side, i.e., to verify the following three equalities:

$$\begin{aligned}(-1)^{p(F)+1}D_\xi^3 F =& 2(-1)^{p(f)+p(g)+1}D_\xi^3 f\cdot D_\xi^3 g\\ \mathrm{Le}_F + A_G =& [\mathrm{Le}_f, A_g] + (-1)^{p(f)p(g)+p(f)+1}\Delta g\cdot A_f\\ A_F =& [A_f, A_g] + 2\cdot(-1)^{p(f)p(g)+p(f)+1}D_\xi^3 g\cdot A_f\end{aligned}$$

The verification is a direct one. □

5.3. Lemma. *The representation of* $i_1 h$ *in the form* (5.1) *is as follows:*

$$i_1 h = \begin{cases} i_2(\Delta(h\xi_1\xi_2\xi_3)) & \text{if } \deg_\xi h = 0,\\ i_2 h + \alpha_{(\Delta h)\xi_1\xi_2\xi_3} & \text{if } \deg_\xi h = 1,\\ \alpha_h & \text{if } \deg_\xi h = 2,\\ \alpha_{\Delta^{-1}(D_\xi^3 h)} & \text{if } \deg_\xi h = 3.\end{cases}$$

Proof. It suffices to compare the definition of α_g with the definitions of i_1 and i_2. If $\deg_\xi h = 0$ use the equalities $\sum \frac{\partial f}{\partial u_i}\xi_j\xi_k = \Delta(f\xi_1\xi_2\xi_3)$ and $A_{\Delta(f\xi_1\xi_2\xi_3)} = \mathrm{Le}_f$. In the remaining cases the verification is not difficult. □

Making use of the Lemmas 5.1, Lemma 5.2 and Lemma 4.2 we can compute the whole multiplication table of $[i_2 f, i_1 h]$:

- $\deg_\xi h = 0$. Then
$$i_1 h = i_2(\Delta(h\xi_1\xi_2\xi_3)) \text{ and } [i_2 f, i_1 h] = i_2\{f, \Delta(h\xi_1\xi_2\xi_3)\}.$$
We also have
$$\{f, \Delta(h\xi_1\xi_2\xi_3)\} = \begin{cases} 0 & \text{if } \deg_\xi f = 3\\ -\{\Delta f, h\}\xi_1\xi_2\xi_3 & \text{if } \deg_\xi f = 2.\end{cases}$$
- $\deg_\xi h = 1$. Then
$$\begin{gathered}[i_2 f, i_1 h] = [i_2 f, i_2 h + \alpha_{(\Delta h)\xi_1\xi_2\xi_3}] =\\ i_2\{f,h\} - i_2(f\Delta h) + i_2(\Delta h\cdot \textstyle\sum \xi_i\partial_{\xi_i} f) + \alpha_{-f\cdot\Delta((\Delta h)\xi_1\xi_2\xi_3)}.\end{gathered}$$
- $\deg_\xi h = 2$. Then
$$\begin{gathered}[i_2 f, i_1 h] = [i_2 f, \alpha_h] = (-1)^{p(f)} i_2(A_h f) - \alpha_{(f\Delta h)} =\\ \begin{cases} i_1(\{f, \Delta h\}\xi_1\xi_2\xi_3) & \text{if } \deg_\xi f = 0\\ i_1(\Delta(fh) - f\Delta h) & \text{if } \deg_\xi f = 1\\ i_2(A_h f) - i_2(\Delta^{-1}D_\xi^3(f\Delta h)) + i_1(\Delta^{-1}D_\xi^3(f\Delta h)) & \text{if } \deg_\xi f = 2\\ -i_2(hD_\xi^3 f) & \text{if } \deg_\xi f = 3.\end{cases}\end{gathered}$$
- $\deg_\xi h = 3$. Then
$$\begin{gathered}[i_2 f, i_1 h] = [i_2 f, \alpha_{\Delta^{-1}(D_\xi^3 h)}] = -\alpha_{f\cdot D_\xi^3 h} =\\ \begin{cases} 0 & \text{if } \deg_\xi f = 0\\ i_1(-\Delta(f\cdot D_\xi^3 h)\xi_1\xi_2\xi_3) = i_1(-f\Delta h - \Delta f\cdot h) & \text{if } \deg_\xi f = 1\\ i_1(-f\cdot D_\xi^3 g) & \text{if } \deg_\xi f = 2\\ i_1(\Delta^{-1}(D_\xi^3 f\cdot D_\xi^3 g)) - i_2(\Delta^{-1}(D_\xi^3 f\cdot D_\xi^3 g)) & \text{if } \deg_\xi f = 3.\end{cases}\end{gathered}$$

The final result is represented in the following tables.

THE BRACKETS $[i_2 f, i_1 h]$

$\deg_\xi(f)$	$\deg_\xi(h) = 0$	$\deg_\xi(h) = 1$
0	$i_2(\{f, \Delta(h\xi_1\xi_2\xi_3)\})$	$-i_1(\{\Delta(f\xi_1\xi_2\xi_3), h\})$
1	$i_2(\{f, \Delta(h\xi_1\xi_2\xi_3)\})$	$i_1(\Delta^{-1}\{f, \Delta h\})+$ $i_2(\{f, h\} - \Delta^{-1}\{f, \Delta h\})$
2	$-i_2(\{\Delta f, h\}\xi_1\xi_2\xi_3)$	$i_2(\Delta(fh) - \Delta(f)h)$
3	0	$i_2(f\Delta(h) + \Delta(f)h)$

$\deg_\xi(f)$	$\deg_\xi(h) = 2$	$\deg_\xi(h) = 3$
0	$i_1(\{f, \Delta h\}\xi_1\xi_2\xi_3)$	0
1	$-i_1(\Delta(fh) + f\Delta h)$	$i_1(-f\Delta(h) - \Delta(f)h)$
2	$i_1(\Delta^{-1}D^3_\xi(f\Delta h))+$ $i_2(A_h f - \Delta^{-1}D^3_\xi(f\Delta h))$	$i_1(-fD^3_\xi h)$
3	$i_2(-hD^3_\xi f)$	$i_1(\Delta^{-1}(D^3_\xi f \cdot D^3_\xi h))-$ $i_2(\Delta^{-1}(D^3_\xi f \cdot D^3_\xi h))$

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